7 C*-Algebras

7.1 The commutative Gelfand-Naimark Theorem

In the same sense as Banach algebras may be seen as an abstraction of the space of continuous operators on a Banach space, we can abstract the concept of continuous operators on a Hilbert space. Of course, a Hilbert space is in particular a Banach space. So the algebras we are looking for are in particular Banach algebras. The additional structure of interest coming from Hilbert spaces is that of an *adjoint*. As in the section about Banach algebras we work in the following exclusively over the field of complex numbers.

Definition 7.1. Let A be an algebra over $\mathbb{C}$. Consider a map $^\star : A \to A$ with the following properties:

- $(a+b)^\star = a^\star + b^\star$ for all $a, b \in A$.
- $(\lambda a)^\star = \overline{\lambda} a^\star$ for all $\lambda \in \mathbb{C}$ and $a \in A$.
- $(ab)^\star = b^\star a^\star$ for all $a, b \in A$.
- $(a^\star)^\star = a$ for all $a \in A$.

Then, * is called an *(anti-linear anti-multiplicative) involution*.

Definition 7.2. Let A be a Banach algebra with involution $^\star : A \to A$ such that $\|a^\star a\| = \|a\|^2$. Then, A is called a *C*-algebra*. For an element $a \in A$, the element $a^\star$ is called its *adjoint*. If $a^\star = a$, then a is called *self-adjoint*. If $a^\star a = a a^\star$, then a is called *normal*.

Exercise 38. Let A be a C*-algebra. (a) Show that $\|a^\star\| = \|a\|$ and $\|aa^\star\| = \|a\|^2$ for all $a \in A$. (b) If $e \in A$ is a unit, show that $e^\star = e$. (c) If $a \in A$ is invertible, show that $a^\star$ is also invertible.

Exercise 39. Let A be a unital C*-algebra and $a \in A$. Show that $\sigma_A(a^\star) = \overline{\sigma_A(a)}$.

Exercise 40. Let X be a Hilbert space. (a) Show that $\mathrm{CL}(X, X)$ is a unital C*-algebra. (b) Show that $\mathrm{KL}(X, X)$ is a C*-ideal in $\mathrm{CL}(X, X)$.

Exercise 41. Let A be a C*-algebra and $a \in A$. Show that there is a unique way to write $a = b + ic$ so that b and c are self-adjoint.

Exercise 42. Let T be a compact topological space. Show that the Banach algebra $\mathrm{C}(T, \mathbb{C})$ of Exercise 26 is a C*-algebra, where the involution is given by complex conjugation.

Proposition 7.3. *Let A be a C*-algebra and $a \in A$ normal. Then,* $\|a^2\| = \|a\|^2$ *and* $r_A(a) = \|a\|$.

Proof. We have $\|a^2\|^2 = \|(a^2)^\star(a^2)\| = \|(a^\star a)^\star(a^\star a)\| = \|a^\star a\|^2 = (\|a\|^2)^2$. This implies the first statement. Also, this implies $\|a^{2^k}\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$ and hence $\lim_{n\to\infty} \|a^n\|^{1/n} = \|a\|$ if the limit exists. But by Proposition 5.12 the limit exists and is equal to $r_A(a)$. □

Proposition 7.4. *Let A be a C*-algebra and $a \in A$ self-adjoint. Then,* $\sigma_A(a) \subset \mathbb{R}$.

Proof. Take $\alpha + \mathrm{i}\beta \in \sigma_A(a)$, where $\alpha, \beta \in \mathbb{R}$. Thus, for any $\lambda \in \mathbb{R}$ we have $\alpha + \mathrm{i}(\beta + \lambda) \in \sigma_A(a + \mathrm{i}\lambda e)$. By Proposition 5.7 we have $|\alpha + \mathrm{i}(\beta + \lambda)| \le \|a + \mathrm{i}\lambda e\|$. We deduce

$$\begin{aligned}
\alpha^2 + (\beta+\lambda)^2 &= |\alpha + \mathrm{i}(\beta+\lambda)|^2 \\
&\le \|a + \mathrm{i}\lambda e\|^2 \\
&= \|(a + \mathrm{i}\lambda e)^\star(a + \mathrm{i}\lambda e)\| \\
&= \|(a - \mathrm{i}\lambda e)(a + \mathrm{i}\lambda e)\| \\
&= \|a^2 + \lambda^2 e\| \\
&\le \|a^2\| + \lambda^2
\end{aligned}$$

Subtracting λ^2 on both sides we are left with $\alpha^2 + \beta^2 + 2\beta\lambda \le \|a^2\|$. Since this is satisfied for all $\lambda \in \mathbb{R}$ we conclude $\beta = 0$. □

Proposition 7.5. *Let A be a unital C*-algebra. Then, the Gelfand transform* $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ *is a continuous unital C*-algebra homomorphism. Moreover, its image is dense in* $\mathrm{C}(\Gamma_A, \mathbb{C})$.

Proof. By Theorem 5.28, the Gelfand transform is a continuous unital algebra homomorphism. We proceed to show that it respects the *-structure. Let $a \in A$ be self-adjoint. Then, combining Proposition 5.27 with Proposition 7.4 we get $\hat{a}(\phi) = \phi(a) \in \sigma_A(a) \subset \mathbb{R}$ for all $\phi \in \Gamma_A$. So $\hat{a}$ is real-valued, i.e., self-adjoint. In particular, $\widehat{a^\star} = \hat{a}^\star$. Using the decomposition of Exercise 41 this follows for general elements of A. **(Explain!)**

It remains to show that the image $\hat{A}$ of the Gelfand transform is dense. It is clear that $\hat{A}$ separates points of Γ_A by construction, vanishes nowhere (as it contains a unit) and is invariant under complex conjugation (as it is the image of a *-algebra homomorphism). Thus, the Stone-Weierstrass Theorem 4.9 ensures that $\hat{A}$ is dense in $\mathrm{C}(\Gamma_A, \mathbb{C})$. □

Theorem 7.6 (Gelfand-Naimark). *Let A be a unital commutative $C^\star$-algebra. Then, the Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is an isometric isomorphism of unital commutative $C^\star$-algebras.*

Proof. Using Proposition 7.5 it remains to show that the Gelfand transform is isometric. Surjectivity then follows from the fact that the isometric image of a complete set is complete and hence closed. Since A is commutative all its elements are normal. Then, by Proposition 7.3, $\|a^2\| = \|a\|^2$ and we can apply Proposition 5.29 to conclude isometry. □

The Gelfand-Naimark Theorem 7.6 (in view of Exercise 29) gives rise to a one-to-one correspondence between compact Hausdorff spaces and unital commutative C*-algebras.

Theorem 7.7. *The category of compact Hausdorff spaces is naturally equivalent to the category of unital commutative $C^\star$-algebras.*

Proof. **Exercise.** □

Before we proceed we need a few more results about C*-algebras.

Lemma 7.8. *Let T_1 be a compact Hausdorff space, T_2 be a Hausdorff space and $f : T_1 \to T_2$ a continuous bijective map. Then, f is a homeomorphism.*

Proof. The image of a compact set under f is compact and hence closed in T_2. But every closed set in T_1 is compact, so f is open and hence a homeomorphism. □

Proposition 7.9. *Let A be a unital $C^\star$-algebra and $a \in A$ normal. Define B to be the unital $C^\star$-subalgebra of A generated by a. Then, B is commutative and the Gelfand transform $\hat{a}$ of a defines a homeomorphism onto its image, $\Gamma_B \to \sigma_B(a)$ which we denote by $\tilde{a}$.*

Proof. B consists of possibly infinite linear combinations of elements of the form $(a^\star)^m a^n$ where $n, m \in \mathbb{N}_0$ (and $a^0 = (a^\star)^0 = e$). In particular, B is commutative. Consider the Gelfand transform $\hat{a} : \Gamma_B \to \mathbb{C}$ of a in B. Suppose $\hat{a}(\phi) = \hat{a}(\psi)$ for $\phi, \psi \in \Gamma_B$. Then, $\phi(a) = \psi(a)$, but also

$$\phi(a^\star) = \hat{a^\star}(\phi) = \overline{\hat{a}(\phi)} = \overline{\hat{a}(\psi)} = \hat{a^\star}(\psi) = \psi(a^\star),$$

using Proposition 7.5. Thus, ϕ is equal to ψ on monomials $(a^\star)^m a^n$ by multiplicativity and hence on all of B by linearity and continuity. This shows that $\hat{a}$ is injective. By Proposition 5.27 the image of $\hat{a}$ is $\sigma_B(a)$. Thus, $\hat{a}$ is a continuous bijective map $\hat{a} : \Gamma_B \to \sigma_B(a)$. With Lemma 7.8 it is even a homeomorphism. □

Proposition 7.10. *Let A be a unital $C^\star$-algebra and $a \in A$. Let B be a unital $C^\star$-subalgebra containing a. Then, $\sigma_B(a) = \sigma_A(a)$.*

Proof. It is clear that $\sigma_A(a) \subseteq \sigma_B(a)$. It remains to show that if $b := \lambda e - a$ for any $\lambda \in \mathbb{C}$ has an inverse in A then this inverse is also contained in B.

Assume first that a (and hence b) is normal. We show that b^{-1} is even contained in the unital C*-subalgebra C of B that is generated by b. Suppose that b^{-1} is not contained in C and hence $0 \in \sigma_C(b)$. Choose $m > \|b^{-1}\|$ and define a continuous function $f : \sigma_C(b) \to \mathbb{C}$ such that $f(0) = m$ and $|f(x)x| \leq 1$ for all $x \in \sigma_C(b)$. Using Theorem 7.6 and Proposition 7.9 there is a unique element $c \in C$ such that $\hat{c} = f \circ \tilde{b}$. Observe also that $\hat{b} = i \circ \tilde{b}$, where $i : \sigma_C(b) \to \mathbb{C}$ is the inclusion map $x \mapsto x$ and hence $\hat{c}\hat{b} = (f \cdot i) \circ \tilde{b}$. Using Theorem 7.6 we find

$$m \leq \|f\| = \|c\| = \|cbb^{-1}\| \leq \|cb\|\|b^{-1}\| = \|f \cdot i\|\|b^{-1}\| \leq \|b^{-1}\|.$$

This contradicts $m > \|b^{-1}\|$. So $0 \notin \sigma_C(b)$ and $b^{-1} \in C$ as was to be demonstrated. This concludes the proof for the case that a is normal.

Consider now the general case. If b is not invertible in B then by Lemma 5.8 at least one of the two elements $b^\star b$ or $bb^\star$ is not invertible in B. Suppose $b^\star b$ is not invertible in B (the other case proceeds analogously). $b^\star b$ is self-adjoint and in particular normal so the version of the proposition already proofed applies and $\sigma_A(b^\star b) = \sigma_B(b^\star b)$. In particular, $b^\star b$ is not invertible in A and hence b cannot be invertible in A. This completes the proof. □

7.2 Spectral decomposition of normal operators

Proposition 7.11 (Spectral Theorem for Normal Elements)**.** *Let A be a unital $C^\star$-algebra and $a \in A$ normal. Then, there exists an isometric homomorphism of unital *-algebras $\phi : \mathrm{C}(\sigma_A(a), \mathbb{C}) \to A$ such that $\phi(\mathbf{1}) = a$.*

Proof. **Exercise.**Hint: Combine Proposition 7.9 with Theorem 7.6. □

Of course, an important application of this is the case when A is the algebra of continuous operators on some Hilbert space and a is a normal operator.

In the context of this proposition we also use the notation $f(a) := \phi(f)$ for $f \in \mathrm{C}(\sigma_A(a), \mathbb{C})$. We use the same notation if f is defined on a larger subset of the complex plane.

Corollary 7.12 (Continuous Spectral Mapping Theorem)**.** *Let A be a unital C*-algebra, $a \in A$ normal and $f : T \to \mathbb{C}$ continuous such that $\sigma_A(a) \subseteq T$. Then, $\sigma_A(f(a)) = f(\sigma_A(a))$.*

Proof. **Exercise.** ☐

Corollary 7.13. *Let A be a unital C*-algebra and $a \in A$ normal. Furthermore, let $f : \sigma_A(a) \to \mathbb{C}$ and $g : f(\sigma_A(a)) \to \mathbb{C}$ continuous. Then $(g \circ f)(a) = g(f(a))$.*

Proof. **Exercise.** ☐

Definition 7.14. Let A be a unital C*-algebra. If $u \in A$ is invertible and satisfies $u^\star = u^{-1}$ we call u *unitary*. If $p \in A$ is self-adjoint and satisfies $p^2 = p$ we call it an *orthogonal projector*. (**Exercise.**Justify this terminology!)

Exercise. Let A be a unital C*-algebra.

1. Let $u \in A$ be unitary. What can you say about $\sigma_A(u)$?

2. Let $p \in A$ be an orthogonal projector. Show that $\sigma_A(p) \subseteq \{0, 1\}$.

3. Let $a \in A$ be normal and $\sigma_A(a) \subset \mathbb{R}$. Show that a is self-adjoint.

Proposition 7.15. *Let A be a unital C*-algebra and $a \in A$ normal. Suppose the spectrum of a is the disjoint union of two non-empty subsets $\sigma_A(a) = s_1 \cup s_2$. Then, there exist $a_1, a_2 \in A$ normal, such that $\sigma_A(a_1) = s_1$ and $\sigma_A(a_2) = s_2$ and $a = a_1 + a_2$. Moreover, $a_1 a_2 = a_2 a_1 = 0$ and a commutes both with a_1 and a_2.*

Proof. **Exercise.** ☐

Proposition 7.16. *Let H be a Hilbert space, $A := \mathrm{CL}(H, H)$ and $k \in \mathrm{KL}(H, H)$ normal. Then, there exists an orthogonal projector $p_\lambda \in A$ for each $\lambda \in \sigma_A(k)$ such that $p_\lambda p_{\lambda'} = 0$ if $\lambda \neq \lambda'$ and*

$$k = \sum_{\lambda \in \sigma_A(k)} \lambda p_\lambda \quad \textit{and} \quad e = \sum_{\lambda \in \sigma_A(k)} p_\lambda .$$

Proof. **Exercise.** (Explain also in which sense the sums converge!) ☐

7.3 Positive elements and states

We now move towards a characterization of noncommutative C*-algebras. We are going to show that any unital C*-algebra is isomorphic to a C*-subalgebra of the algebra of continuous operators on some Hilbert space.

Definition 7.17. Let A be a unital C*-algebra. A self-adjoint element $a \in A$ is called *positive* iff $\sigma_A(a) \subset [0, \infty)$.

Exercise 43. Let T be a compact Hausdorff space and consider the C*-algebra $\mathrm{C}(T, \mathbb{C})$. Show that the self-adjoint elements are precisely the real valued functions and the positive elements are the functions with non-negative values.

Proposition 7.18. *Let A be a unital C*-algebra and $a, b \in A$ positive. Then, $a + b$ is positive.*

Proof. Suppose $\lambda \in \sigma_A(a + b)$. Since a and b are self-adjoint so is $a + b$. In particular, $\sigma_A(a + b) \subset \mathbb{R}$ and λ is real. Set $\alpha := \|a\|$ and $\beta := \|b\|$. Then, $(\alpha+\beta)-\lambda \in \sigma_A((\alpha+\beta)e-(a+b))$ and thus $|(\alpha+\beta)-\lambda| \le r_A((\alpha+\beta)e-(a+b))$ by Theorem 5.14. But the element $(\alpha + \beta)e - (a + b)$ is normal (and even self-adjoint), so Proposition 7.3 applies and we have $r_A((\alpha+\beta)e-(a+b)) = \|(\alpha+\beta)e-(a+b)\| \le \|\alpha e - a\| + \|\beta e - b\|$. Again using Proposition 7.3 we find $\|\alpha e - a\| = r_A(\alpha e - a)$ and $\|\beta e - b\| = r_A(\beta e - b)$. But $\sigma_A(a) \subseteq [0, \alpha]$ by positivity and Proposition 5.7. Thus, $\sigma_A(\alpha e - a) \subseteq [0, \alpha]$. Hence, by Theorem 5.14, $r_A(\alpha e - a) \le \alpha$. In the same way we find $r_A(\beta e - b) \le \beta$. We have thus demonstrated the inequality $|(\alpha + \beta) - \lambda| \le \alpha + \beta$. This implies $\lambda \ge 0$, completing the proof. □

Proposition 7.19. *Let A be a unital C*-algebra and $a \in A$ self-adjoint. Then, there exist positive elements $a_+, a_- \in A$ such that $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.*

Proof. **Exercise.** Hint: Consider the unital C*-subalgebra generated by a. □

Proposition 7.20. *Let A be a unital C*-algebra and $a \in A$. Then, a is positive iff there exists $b \in A$ such that $a = b^\star b$.*

Proof. **Exercise.** □

Lemma 7.21. *Let A be a unital C*-algebra and $a \in A$ positive and such that $\|a\| \le 1$. Then, $e - a$ is positive and $\|e - a\| \le 1$.*

Proof. **Exercise.** □

A similar role to that played by the characters in the theory of commutative C⋆-algebras is now played by *states*.

Definition 7.22. Let A be a unital C⋆-algebra. A continuous linear functional $\omega : A \to \mathbb{C}$ is called *positive* iff $\omega(a) \geq 0$ for all positive elements $a \in A$. A positive functional $\omega : A \to \mathbb{C}$ is called a *state* iff it is normalized, i.e., iff $\|\omega\| = 1$. The set Σ_A of states of A is called the *state space* of A.

Exercise 44. Let A be a unital C⋆-algebra. Show that $\Gamma_A \subseteq \Sigma_A$, i.e., each character is in particular a state.

Proposition 7.23. *Let A be a unital C⋆-algebra and ω a positive functional on A. Then $\omega(a^\star) = \overline{\omega(a)}$ for all a in A. In particular, $\omega(a) \in \mathbb{R}$ if a is self-adjoint.*

Proof. **Exercise.** □

Proposition 7.24. *Let A be a unital C⋆-algebra and ω a positive functional on A. Consider the map $[\cdot,\cdot]_\omega : A \times A \to \mathbb{C}$ given by $[a,b]_\omega = \omega(b^\star a)$. It has the following properties:*

1. *$[\cdot,\cdot]_\omega$ is a sesquilinear form on A.*
2. *$[a,b]_\omega = \overline{[b,a]_\omega}$ for all $a,b \in A$.*
3. *$[a,a]_\omega \geq 0$ for all $a \in A$.*

Proof. **Exercise.** □

This shows that we almost have a scalar product, only the definiteness condition is missing. Nevertheless we have the Cauchy-Schwarz inequality.

Proposition 7.25. *Let A be a unital C⋆-algebra and ω a non-zero positive functional on A. The following is true:*

1. *$|[a,b]_\omega|^2 \leq [a,a]_\omega [b,b]_\omega$ for all $a,b \in A$.*
2. *Let $a \in A$. Then, $[a,a]_\omega = 0$ iff $[a,b]_\omega = 0$ for all $b \in A$.*
3. *$[ab,ab]_\omega \leq \|a\|^2 [b,b]_\omega$ for all $a,b \in A$.*

Proof. **Exercise.** □

Proposition 7.26. *Let A be a unital $C^\star$-algebra and $\omega : A \to \mathbb{C}$ continuous and linear. Then, ω is a positive functional iff $\|\omega\| = \omega(e)$.*

Proof. Suppose that ω is a positive functional. Given $\epsilon > 0$ there exists $a \in A$ with $\|a\| = 1$ such that $\|\omega(a)\|^2 \geq \|\omega\|^2 - \epsilon$. Using the Cauchy-Schwarz inequality (Proposition 7.25.1) with $b = e$ we find

$$\|\omega(a)\|^2 \leq \omega(a^\star a)\omega(e) \leq \|\omega\|\|a^\star a\|\omega(e) = \|\omega\|\omega(e).$$

Combining this with the first inequality we get $\|\omega\|^2 - \epsilon \leq \|\omega\|\omega(e)$. Since ϵ was arbitrary this implies $\|\omega\| \leq \omega(e)$. On the other hand, the inequality $\omega(e) \leq \|\omega\|$ is clear.

Conversely, suppose now that ω is a continuous linear functional with the property $\|\omega\| = \omega(e)$. Without loss of generality we normalize ω such that $\omega(e) = 1 = \|\omega\|$. We first show that $\omega(a) \in \mathbb{R}$ if $a \in A$ is self-adjoint. Assume the contrary, i.e., assume there exists $a \in A$ such that $\omega(a) = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ and $y \neq 0$. Set $b := a - xe$. Then, b is self-adjoint and $\omega(b) = \mathrm{i}y$. For $\lambda \in \mathbb{R}$ we get,

$$|\omega(b + \mathrm{i}\lambda e)|^2 = |\mathrm{i}y + \mathrm{i}\lambda\omega(e)|^2 = y^2 + 2\lambda y + \lambda^2.$$

One the other hand,

$$|\omega(b + \mathrm{i}\lambda e)|^2 \leq \|\omega\|^2\|b + \mathrm{i}\lambda e\|^2 = \|(b + \mathrm{i}\lambda e)^\star(b + \mathrm{i}\lambda e)\| \leq \|b\|^2 + \lambda^2.$$

The resulting inequality is equivalent to,

$$y^2 + 2\lambda y \leq \|b\|^2,$$

which obviously cannot be fulfilled for arbitrary $\lambda \in \mathbb{R}$ (recall that $y \neq 0$), giving a contradiction. This shows that $\omega(a) \in \mathbb{R}$ if $a \in A$ is self-adjoint.

We proceed to show that $\omega(a) \geq 0$ if $a \in A$ is positive. Assume the contrary, i.e., assume there is $a \in A$ positive such that $\omega(a) < 0$. (Note that $\omega(a) \in \mathbb{R}$ by the previous part of the proof.) By suitable normalization we can achieve $\|a\| \leq 1$ as well. By Lemma 7.21 we have $\|e - a\| \leq 1$ and thus $|\omega(e - a)| \leq 1$ since $\|\omega\| = 1$. On the other hand, $|\omega(e - a)| = |1 - \omega(a)| > 1$, a contradiction. This shows that ω must be positive. □

Proposition 7.27. *Let A be a unital $C^\star$-algebra and $a \in A$ positive. Then, there exists a state $\omega \in \Sigma_A$ such that $\omega(a) = \|a\|$.*

Proof. Since a is positive we have $\sigma_A(a) \subseteq [0, \infty)$. Moreover, a is normal, so by Proposition 7.3 we have $r_A(a) = \|a\|$. Thus, $\|a\| \in \sigma_A(a)$. Let B be the unital C*-subalgebra of A generated by a. By Proposition 7.10 we have $\sigma_B(a) = \sigma_A(a)$ and in particular $\|a\| \in \sigma_B(a)$. By Proposition 7.9, $\hat{a}$ induces a homeomorphism $\Gamma_B \to \sigma_B(a)$. In particular, there exists a character $\phi \in \Gamma_B$ such that $\|a\| = \hat{a}(\phi) = \phi(a)$. Recall that $\phi(e) = 1$ and $\|\phi\| = 1$ by Proposition 5.23. By the Hahn-Banach Theorem (Corollary 3.38) there exists an extension of ϕ to a linear functional $\omega : A \to \mathbb{C}$ such that $\omega|_B = \phi$ and $\|\omega\| = 1$. Note in particular that $\omega(e) = 1 = \|\omega\|$. So by Proposition 7.26, $\omega \in \Sigma_A$. □

7.4 The GNS construction

Proposition 7.28. *Let A be a unital C*-algebra and ω a state on A. Define $I_\omega := \{a \in A : [a, a]_\omega = 0\} \subseteq A$. Then, I_ω is a left ideal of the algebra A. In particular, the quotient vector space A/I_ω is an inner product space with the induced sesquilinear form.*

Proof. **Exercise.** □

Definition 7.29. Let A be a unital C*-algebra and ω a state on A. We call the completion of the inner product space A/I_ω the *Hilbert space associated with the state* ω and denote it by H_ω. We denote its scalar product by $\langle \cdot, \cdot \rangle_\omega : H_\omega \times H_\omega \to \mathbb{C}$.

A consequence of the fact that A/I_ω is a left ideal is that we have a representation of A on this space and its completion from the left.

Definition 7.30. Let A be a unital C*-algebra and H a Hilbert space. A homomorphism of unital *-algebras $A \to \mathrm{CL}(H, H)$ is called a *representation of* A. A representation that is injective is called *faithful*. A representation that is surjective is called *full*.

Proposition 7.31. *Let A, B be unital C*-algebras and $\phi : A \to B$ a homomorphism of unital *-algebras.*

1. *$\|\phi(a)\| \leq \|a\|$ for all $a \in A$. In particular, ϕ is continuous.*
2. *If ϕ is injective then it is isometric.*

Proof. **Exercise.** □

Theorem 7.32. *Let A be a unital $C^\star$-algebra and ω a state on A. Then, there is a natural representation $\pi_\omega : A \to \mathrm{CL}(H_\omega, H_\omega)$. Moreover,*

$$\|\pi_\omega(a)\|^2 \geq \omega(a^\star a) \quad \forall a \in A,$$

and $\|\pi_\omega\| = 1$.

Proof. Define the linear maps $\tilde{\pi}_\omega(a) : A/I_\omega \to A/I_\omega$ by left multiplication, i.e., $\tilde{\pi}_\omega(a) : [b] \mapsto [ab]$. That $\tilde{\pi}_\omega(a)$ is well defined follows from Proposition 7.28 (I_ω is a left ideal). By definition we have then $\tilde{\pi}_\omega(ab) = \tilde{\pi}_\omega(a)\circ\tilde{\pi}_\omega(b)$ and $\tilde{\pi}_\omega(e) = \mathbf{1}_{A/I_\omega}$. Furthermore, $\|\tilde{\pi}_\omega(a)\| \leq \|a\|$ due to Proposition 7.25.3 and hence $\tilde{\pi}_\omega(a)$ is continuous. So we have a homomorphism of unital algebras $\tilde{\pi}_\omega : A \to \mathrm{CL}(A/I_\omega, A/I_\omega)$. Also, $\tilde{\pi}_\omega$ preserves the *-structure because,

$$\langle \tilde{\pi}_\omega(a^\star)[b], [c]\rangle_\omega = [a^\star b, c]_\omega = \omega(c^\star a^\star b) = [b, ac]_\omega = \langle [b], \tilde{\pi}_\omega(a)[c]\rangle_\omega.$$

Since $\tilde{\pi}_\omega(a)$ is continuous it extends to a continuous operator $\pi_\omega(a) : H_\omega \to H_\omega$ on the completion H_ω of A/I_ω, with the same properties. In particular, π_ω is a homomorphism of unital *-algebras.

Due to the bound $\|\tilde{\pi}_\omega(a)\| \leq \|a\|$ and hence $\|\pi_\omega(a)\| \leq \|a\|$ (or due to Proposition 7.31.1) we find $\|\pi_\omega\| \leq 1$. Observe also that $\omega(e) = 1$ By Proposition 7.26 and hence $\|\pi_\omega(a)\|^2 \geq [ae, ae]_\omega/[e, e]_\omega = \omega(a^\star a)$. In particular, $\|\pi_\omega\| \geq \|\pi_\omega(e)\| \geq 1$. Thus, $\|\pi_\omega\| = 1$. □

The construction leading to the Hilbert spaces H_ω and this representation is called the *GNS-construction* (Gelfand-Naimark-Segal).

Definition 7.33. Let A be a unital C*-algebra, H a Hilbert space and $\phi : A \to \mathrm{CL}(H, H)$ a representation. A vector $\psi \in H$ is called a *cyclic vector* iff $\{\phi(a)\psi : a \in A\}$ is dense in H. The representation is then called a *cyclic representation*.

Proposition 7.34. *Let A be a unital $C^\star$-algebra and ω a state on A. Then, there is a cyclic vector $\psi \in H_\omega$ with the property $\omega(a) = \langle \pi_\omega(a)\psi, \psi\rangle_\omega$ for all $a \in A$.*

Proof. **Exercise.** □

A deficiency of the representation of Theorem 7.32 is that it is neither faithful nor full in general. Lack of faithfulness can be remedied. The idea is that we take the direct sum of the representations π_ω for all normalized states ω.

Proposition 7.35. *Let $\{H_\alpha\}_{\alpha \in I}$ be a family of Hilbert spaces. Consider collections ψ of elements $\psi_\alpha \in H_\alpha$ with $\alpha \in I$ such that $\sup_{J \subseteq I} \sum_{\alpha \in J} \|\psi_\alpha\|^2 < \infty$ where J ranges over all finite subsets of I. Then, the set H of such collections ψ is naturally a Hilbert space and we have isometric embeddings $H_\alpha \to H$ for all $\alpha \in I$.*

Proof. **Exercise**. □

Definition 7.36. The Hilbert space H constructed in the preceding Proposition is called the *direct sum* of the Hilbert spaces H_α and is denoted $\bigoplus_{\alpha \in I} H_\alpha$.

Proposition 7.37. *Let A be a unital C*-algebra, $\{H_\alpha\}_{\alpha \in I}$ a family of Hilbert spaces and $\phi_\alpha : A \to \mathrm{CL}(H_\alpha, H_\alpha)$ a representation for each $\alpha \in I$. Then, there exists a representation $\phi : A \to \mathrm{CL}(H, H)$ such that $\|\phi(a)\| = \sup_{\alpha \in I} \|\phi_\alpha(a)\|$ for all $a \in A$, where $H := \bigoplus_{\alpha \in I} H_\alpha$.*

Proof. **Exercise**. □

We are now ready to put everything together.

Theorem 7.38 (Gelfand-Naimark)**.** *Let A be a unital C*-algebra. Then, there exists a Hilbert space H and a faithful representation $\pi : A \to \mathrm{CL}(H, H)$.*

Proof. **Exercise**. □

This result concludes our characterization of the structure of C*-algebras: Each C*-algebra arises as a C*-subalgebra of the algebra of continuous operators on some Hilbert space.

Exercise 45. Let A be a unital C*-algebra, H_1, H_2 Hilbert spaces, $\phi_1 : A \to \mathrm{CL}(H_1, H_1)$ and $\phi_2 : A \to \mathrm{CL}(H_2, H_2)$ cyclic representations. Suppose that $\langle \phi_1(a)\psi_1, \psi_1\rangle_1 = \langle \phi_2(a)\psi_2, \psi_2\rangle_2$ for all $a \in A$, where ψ_1, ψ_2 are the cyclic vectors in H_1 and H_2 respectively. Show that there exists a unitary operator (i.e., an invertible linear isometry) $W : H_1 \to H_2$ such that $\phi(a) = W^\star \psi(a) W$ for all $a \in A$.